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Goldstone bosons and a dynamical Higgs field

Sander Mooij* **and Marieke Postma**†

Nikhef, Science Park 105, 1098 XG Amsterdam, The Netherlands.

Abstract: Higgs inflation uses the gauge variant Higgs field as the inflaton. During inflation the Higgs field is displaced from its minimum, which results in associated Goldstone bosons that are apparently massive. Working in a minimally coupled $U(1)$ toy model, we use the closed-time-path formalism to show that these Goldstone bosons do contribute to the one-loop effective action. Therefore the computation in unitary gauge gives incorrect results. Our expression for the effective action is gauge invariant upon using the background equations of motion.

*smooij@nikhef.nl

†mpostma@nikhef.nl

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1. Introduction

The mechanism of Higgs inflation is already an old idea [1], which was recently revived by Bezrukov and Shaposhnikov [2, 3, 4]. It is elegant in its simplicity: why look for exotic inflatons if the Standard Model already possesses a viable candidate? Inflation is obtained by introducing an additional coupling between the Higgs field and the Ricci scalar $\mathcal{R}$. It offers the exciting possibility that the Higgs mass can be predicted from cosmological data on the cosmic microwave background (CMB) [5, 6]. This requires the computation of the quantum corrections to the potential [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. In this work we want to clarify the role played by Goldstone bosons in the loop calculation.

During inflation and the reheating period afterwards, the Higgs field is evolving in its potential. This complicates the calculation of the one-loop effective action compared to the

vacuum calculation, with the Higgs field in the minimum of its potential. First of all, the Goldstone bosons are now apparently massive and, as we will show, do contribute to the one-loop effective action (see also [7, 8]). Second, there are time-dependent corrections to the Coleman-Weinberg expression which strictly only applies to the static case. Both effects have not been fully appreciated in the literature as they are small during inflation. However, they are important afterwards, and should be taken into account if one wants to relate low energy observables (the Higgs mass to be measured at the LHC) to high energy (CMB) observables.

Goldstone's theorem states that there is one massless boson for each generator of a continuous symmetry that is broken spontaneously by the ground state. In a gauge theory these Goldstone bosons do not appear as independent physical particles. They are "eaten" by the gauge bosons; their associated degree of freedom (d.o.f.) is used to turn a massless vector boson (2 d.o.f.) into a massive one (3 d.o.f.). This is best seen in unitary gauge, in which the Goldstone bosons explicitly disappear from the theory.

During the cosmological evolution of the Higgs field this picture changes. The Higgs field is displaced from its minimum, and is evolving in time. The gauge symmetry is broken, but the associated Goldstone bosons are no longer massless eigenstates. They can still be removed from the theory by going to unitary gauge, (though only upon using the equations of motion). Therefore one might be inclined to think that the Goldstone bosons are still unphysical, and that their contribution to any quantum corrections should be omitted. This would be dramatic for supersymmetric Higgs inflation [21, 22, 23], as the quadratic corrections would no longer cancel.

Potential problems with calculating quantum corrections in unitary gauge were noted before in the literature [24, 25]. To investigate the effect of the massive Goldstone bosons, we use the closed-time-path formalism [26, 27, 28, 29, 30, 31, 32, 33] to compute Coleman-Weinberg one-loop corrections. In this work we restrict ourselves to a minimally coupled U(1) toy model in flat spacetime. *We find that corrections induced by the U(1) Goldstone boson are real and can not be omitted.* Our results apply to Standard Model Higgs inflation, as well as to models in which the inflaton is a Higgs field of some grand unified theory [34] [35]. In addition, we calculate the corrections due to the time-dependence of the Higgs field. These are essential for showing that our result is gauge invariant.

A large part of our computation follows the work by Heitmann and Baacke [36, 37, 38, 39, 40, 41, 42]. We generalize their results for an arbitrary Higgs potential. We calculate the equation of motion for the background field rather than the effective action directly; up to a field-independent constant the latter can always be obtained by formally integrating the field equations. Our results reduce to the original Coleman-Weinberg result in the static limit. Our calculation is done in R_ξ gauge. Boyanovsky et al. have calculated the one-loop potential in terms of gauge invariant quantities [43], but only in the adiabatic limit, which does not take into account the time-dependence of the rolling Higgs field.

We will be working in Minkowski spacetime, with $\{+ - - -\}$ signature, and set $\hbar = c = k_B = 1$. We choose Feynman-'t Hooft gauge $\xi = 1$. In the appendix we calculate the equation of motion perturbatively, in arbitrary R_ξ gauge. There we show that the gauge-dependent

terms cancel upon using the equation of motion for the background field ϕ. The effective potential has already been shown to be gauge invariant when calculated around a potential minimum [24, 44]. Here we show that gauge invariance holds also in this more general case at the one-loop level, but only on-shell, upon using the background equations of motion.

The article is organized as follows. In the next section we discuss the Abelian Higgs model at the classical level. We start by generalizing Goldstone's theorem to the case with the Higgs field displaced from its minimum, relating the Goldstone boson mass to the slope of the Higgs potential. Although apparently massive, the Goldstone bosons can still be removed from the theory in unitary gauge, but only upon using the equations of motion. We end the section with a discussion of the problems encountered if one attempts to calculate the one-loop effective action in unitary gauge [24, 25]. To resolve these problems we calculate the equations of motion in section 3. The calculation is set up in a non-perturbative way. However, to extract the divergent parts explicitly, we use a perturbative expansion. We end with a discussion of our results in section 4. A brief outline of the CTP formalism, and our definitions and conventions used, are relegated to appendix A. In appendix B we present a perturbative calculation of the equations of motion in arbitrary R_ξ gauge. Although more technically involved, it shows explicitly that the results are gauge invariant upon using the background equation of motion.

2. The rolling Goldstone boson

In this section we show how the usual Goldstone boson theorem [45, 46] changes when we consider a global U(1) symmetry broken by a scalar field that is *not* in its minimum. We then discuss how this affects the Higgs mechanism in the gauged version of the theory. It still seems possible to go to unitary gauge. However, studying the associated Coleman-Weinberg corrections suggests a problem with this gauge. For simplicity, we will focus on a U(1) gauge theory. The results can be easily generalized to non-Abelian gauge groups.

2.1 Goldstone boson theorem

Consider a theory with a complex scalar field Φ, which we will refer to as the Higgs field. It is invariant under a global U(1) transformation. The field has a time-dependent expectation value $\Phi_{\rm cl} = (\phi_R(t) + i\phi_I(t))/\sqrt{2}$; without loss of generality we can align this with the real direction and set $\phi_I = 0$. Goldstone showed that in the broken phase $\phi_R \neq 0$ there is a massless excitation in the spectrum, provided the potential is extremized [45, 46]. Here we repeat his argument for a (time-dependent) classical background field which is displaced from its minimum $\partial_{\phi_R} V|_{\rm cl} \neq 0$.

Under an infinitesimal global U(1) transformation $\Phi \to {\rm e}^{i\alpha}\Phi$ the invariant potential $V(\Phi\Phi^\dagger)$ transforms as

$$\delta_\alpha V = \frac{\partial V}{\partial \phi_i}\delta_\alpha \phi_i = 0, \tag{2.1}$$

with $i = \{R, I\}$. Written out in terms of real fields the change under a gauge transformation is $\delta_\alpha \phi_R = -\alpha \phi_I$ and $\delta_\alpha \phi_I = \alpha \phi_R$. Differentiating (2.1) with respect to ϕ_k, the equation for $k = R$ is trivially satisfied. For $k = I$ evaluated on the classical background configuration it yields, however,

$$\frac{\partial^2 V}{\partial \phi_I \partial \phi_I} \phi_R - \left. \frac{\partial V}{\partial \phi_R} \right|_{\rm cl} = 0. \tag{2.2}$$

If the Higgs extremizes the potential, the second term in the equation above vanishes. One concludes that the spectrum contains a massless Goldstone boson. However, with the Higgs displaced from its minimum — as is the case during Higgs inflation — the first derivative of the potential no longer vanishes. Therefore the Goldstone boson mass is apparently non-zero:

$$m_I^2 \equiv \left. \frac{\partial^2 V}{\partial \phi_I^2} \right|_{\rm cl} = \left. \frac{1}{\phi_R} \frac{\partial V}{\partial \phi_R} \right|_{\rm cl} = -\left. \frac{\ddot{\phi}_R}{\phi_R} \right|_{\rm cl}. \tag{2.3}$$

Strictly speaking, we can only unambiguously identify the mass of excited states with the (eigenstates of the) second derivative of the potential if the potential is minimized. In a time-dependent background fields may mix non-trivially in the kinetic terms as well. Throughout the paper we will be sloppy with this distinction and equally use "mass matrix" and "second derivative of the potential" $m_{ij} \equiv V_{\phi_i \phi_j}$, as was done in (2.3) above. The last equality is only valid on-shell, as we used that the evolution of the classical background $\phi_R(t)$ is governed by the Klein-Gordon equation, which in a Minkowski universe reads $\ddot{\phi}_R + \partial_{\phi_R} V = 0$.

2.2 Higgs mechanism

We now gauge the U(1) model of the previous section. How does the Higgs mechanism work during inflation, when the Higgs is displaced from its minimum and the Goldstone boson is massive? The standard lore found in textbooks is that the gauge boson cannot obtain a mass, unless this mass term is associated with a pole in the vacuum polarization amplitude, which can only be created by a massless scalar particle.

The Lagrangian of the U(1) Abelian Higgs model is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi (D^\mu \Phi)^\dagger - V(\Phi \Phi^\dagger), \tag{2.4}$$

with $F_{\mu\nu}$ the Abelian field strength and $D_\mu \Phi = (\partial_\mu + i g A_\mu)\Phi$ the covariant derivative. Under a $U(1)$ gauge transformation the Higgs and gauge field transform

$$\Phi \to {\rm e}^{i\alpha} \Phi, \qquad A_\mu \to A_\mu - \frac{1}{g} \partial_\mu \alpha, \tag{2.5}$$

with α the infinitesimal parameter of the gauge transformation, and g the U(1) gauge coupling. To analyze the Higgs mechanism we perturb the Higgs field around the classical background:

$$\begin{aligned} \Phi(x,t) &= \frac{1}{\sqrt{2}} \left(\Phi_R(x,t) + i \Phi_I(x,t) \right) = \frac{1}{\sqrt{2}} \big[(\phi_R(t) + h(x,t)) + i\theta(x,t) \big], \\ A_\mu(x,t) &= A_\mu(x,t), \end{aligned} \tag{2.6}$$

with as before $\phi_R(t)$ the classical background field, and $h(x,t)$, $\theta(x,t)$, $A_\mu(x,t)$ the fluctuations of the Higgs and gauge field respectively.

The potential $V(\Phi\Phi^\dagger)$ can be expanded in the perturbed fields

$$V = V|_{\rm cl} + V_R|_{\rm cl}\, h + \frac{1}{2} V_{RR}|_{\rm cl}\, h^2 + \frac{1}{2} V_{II}|_{\rm cl}\, \theta^2 + ... \tag{2.7}$$

with the dots representing terms of cubic order or higher in the fluctuations. Here we introduced the notation $V_i = \partial_{\Phi_i} V$. Because of the U(1) symmetric form of the potential there are no terms linear in θ. There is however a tadpole term in h if the Higgs is displaced from its minimum. Similarly we expand the kinetic terms:

$$\begin{aligned}\mathcal{L}_{\rm kin} =& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu h \partial^\mu h + \partial_\mu \theta \partial^\mu \theta + g^2 \phi_R^2 A_\mu A^\mu\right) + g\phi_R A_\mu \partial^\mu \theta \\ & - g\dot\phi_R \theta A_0 + \dot h \dot\phi_R + \frac{1}{2}\dot\phi_R^2 + \ldots \end{aligned}\tag{2.8}$$

The terms in the second line are absent for a Higgs field in a static minimum.

Now we transform to unitary gauge. Define a new gauge field via

$$A_\mu = B_\mu - \frac{1}{g}\partial_\mu(\theta/\phi_R). \tag{2.9}$$

This leaves the potential and the kinetic term for the gauge fields invariant, but affects the Higgs kinetic terms. Writing the kinetic Lagrangian in terms of the newly defined field B_μ removes the kinetic term for the Goldstone θ and its derivative coupling to the gauge field:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2}\left(\partial_\mu h \partial^\mu h + g^2 \phi_R^2 B_\mu B^\mu\right) - \frac{\theta^2 \dot\phi_R^2}{2\phi_R^2} + \frac{\theta\dot\theta\dot\phi_R}{\phi_R} + \dot h \dot\phi_R + \frac{1}{2}\dot\phi_R^2 + \ldots \tag{2.10}$$

where $B_{\mu\nu}$ is the Abelian field strength for B_μ. If the potential is minimized we have $V_R = 0$ and $\dot\phi_R = 0$, as in the usual description of the Higgs mechanism. The Goldstone boson completely disappears from the Lagrangian. It is eaten by the longitudinal component of the gauge field A_L which has become massive: $m_A = g\phi_R$. However, with the Higgs displaced from its minimum, the Goldstone boson cannot be eliminated from the Lagrangian by the field redefinition (2.9), or equivalently by a unitary gauge transformation (2.5) with $\alpha = \theta/\phi_R$. The θ-field is still present, both in the kinetic and in the potential part of the Lagrangian. Nevertheless, the gauge field has still become massive. How is this possible without a massless pole in the polarization tensor? The answer lies in the last four time-dependent terms in (2.10). These exactly cancel the Goldstone mass term in (2.7) when the fields are taken on-shell. Indeed

$$\begin{aligned}\mathcal{L}_{\rm kin} \supset& -\frac{\theta^2 \dot\phi_R^2}{2\phi_R^2} + \frac{\theta\dot\theta\dot\phi_R}{\phi_R} + \dot h \dot\phi_R + \frac{1}{2}\dot\phi_R^2 = -\frac{\ddot\phi_R}{2}\left(\frac{\theta^2}{\phi_R} + 2h + \phi_R\right) \\ =& \frac{1}{2} V_{II}|_{\rm cl}\,(\theta^2 + 2\phi_R h + \phi_R^2). \end{aligned}\tag{2.11}$$

To get the second expression we used partial integration, whereas to obtain the final result we used the generalized Goldstone theorem (2.3), which follows from gauge invariance and the background equations of motion. The first term in (2.11) exactly cancels the mass term V_{II} in the potential (2.7). Hence, taking the system on-shell, all θ-dependent terms can be eliminated, and in this sense it is still possible to go to unitary gauge. The gauge field acquires a mass by eating the massless Goldstone. The second term in (2.11) cancels the tadpole in the potential. This just reflects that even though ϕ_R does not minimize the potential, on-shell it does extremize the action, and thus $\delta S/\delta\phi_R = 0$. Finally the last term just contributes to the background energy density, which gets contributions from both kinetic and potential terms.

Finally we remark that we used the decomposition (2.6) merely for its computational advantages in the next section. To see that the Goldstone boson θ disappears from the action in unitary gauge, it is easier to use the decomposition $\Phi = \rho e^{i\theta}$. Here one does not even need to go on-shell to see the Goldstone boson θ disappear from the action in unitary gauge.

2.3 Coleman-Weinberg corrections

For a theory described by a set of quantum fields of spin J_i Coleman and Weinberg (CW) have calculated the one-loop corrections to the effective action $\Gamma^{1-\text{loop}} = \int \mathrm{d}^4x V_{\text{CW}}$, with [47][1]

$$\begin{aligned} V_{\text{CW}} &= \frac{1}{2}\sum_i (-1)^{2J_i}(2J_i+1)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2+m_i^2} \\ &= \frac{1}{16\pi^2}\sum_i (-1)^{2J_i}(2J_i+1)\left(m_i^2\Lambda^2 - \frac{1}{4}m_i^4 \ln\left(\frac{\Lambda^2}{m^2}\right) + \dots\right). \end{aligned} \tag{2.12}$$

Here the sum is over all the fields in the model and Λ denotes the energy cut-off. We only kept the relevant (divergent and field-dependent) terms. This expression is valid for a theory in which the Higgs field is in its minimum, and the background is time-independent. We now wish to find out how to calculate CW corrections for an evolving Higgs field. Based on the discussion in this section, we would be tempted to use unitary gauge. If we take the system on-shell, all reference to the Goldstone boson mass can be eliminated. The CW potential is then obtained by summing over the real part of the Higgs field h, and the massive gauge boson. This procedure, however, leads to problems.

First of all, in a globally supersymmetric theory there are no quadratic divergences in (2.12), as the bosonic and fermionic contributions cancel out. However, here one calculates masses as second derivatives of the Lagrangian, without demanding the background Higgs field to be on-shell. Hence, this calculation also takes into account a non-zero $m_I^2 = V_{II}|_{\text{cl}}$. If we remove the Goldstone boson "by hand" by going to unitary gauge, this implies removing the non-zero term in (2.12) corresponding to $m_I \neq 0$. Consequently, the quadratic divergences

[1] Note that the cutoff in this expression is on spatial momenta, which explains the difference in coefficients with the more common expression with a cutoff on Euclidean 4-momentum.

would no longer vanish. If true, this would have huge consequences for supersymmetric cosmology. For example, it would be disastrous for supersymmetric Higgs inflation [21, 22, 23].

A related problem with removing the Goldstone boson "by hand" is that it gives a discontinuous one-loop potential. When the Higgs field moves from $\phi_R = 0$ to an infinitesimally small amount $\phi_R = \epsilon$, we go from the symmetric to the broken phase. The d.o.f. in the symmetric phase are the real and imaginary parts of the Higgs h and θ, whereas in the broken phase in unitary gauge we only have the Higgs h and the massive gauge boson A_μ. Suddenly the Goldstone boson θ would not be physical anymore. Its contribution to the Coleman-Weinberg potential, therefore, should be omitted, causing a discontinuity in the potential. This cannot be correct.

Therefore we should calculate the Coleman-Weinberg potential 2.12 for a Higgs field displaced from its minimum, in a gauge different from unitary gauge, and check whether it indeed makes sense to simply omit the Goldstone boson.

3. Non-perturbative calculation of the equations of motion

The previous section's considerations lead us to a careful analysis of the Coleman-Weinberg corrections to a theory with a displaced Higgs field. To take the time-dependence into account we use the Schwinger-Keldysh or closed-time-path (CTP) formalism [26, 27, 28, 29, 30, 31, 32, 33]. In this formalism one compares two *in*-states rather than an *in*-state and an *out*-state. As we are interested in expectation values at one given point in time, not in transition amplitudes, it seems more useful to work in this formalism where we do not need to know the *out*-state explicitly. More details on the CTP formalism can be found in appendix A. As it turns out, the difference between the CTP and the usual S-matrix approach in the non-perturbative one-loop calculation discussed below vanishes, and no specific CTP knowledge is needed. This is different for the perturbative one-loop calculation presented in appendix B. Our notation and calculation closely follow the work of Heitmann and Baacke [36, 37, 38, 39, 40, 41, 42].

Rather then determining the quantum effective action, it turns out easier to calculate the one-loop corrected equations of motion for the classical field. The reason is that the latter can be expressed directly in terms of the resummed propagator, and as such allows for a non-perturbative approach. The equations of motion follow from the effective action Γ in the CTP formalism via $\delta\Gamma/\delta\phi_+|_{\{J_+=J_-=0\}} = 0$. Hence, up to a field-independent constant the effective action can always be obtained by formally integrating the field equations.

3.1 Gauge fixing

To gauge fix the action we use R_ξ-gauge. We add a gauge fixing term

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}G^2, \qquad G = \partial_\mu A^\mu - \xi g(\phi + h)\theta. \tag{3.1}$$

For notational convenience we dropped the subscript R from the classical background field. With this choice the term $\propto A^\mu\partial_\mu\theta(\phi + h)$ in the kinetic terms (2.8) is eliminated. The

corresponding Faddeev-Popov determinant is

$$\mathcal{L}_{\rm FP} = \bar{\eta} g \frac{\delta G}{\delta \alpha} \eta = \bar{\eta} \left[-\partial^2 - \xi g^2 (\phi + h)^2 + \xi g^2 \theta^2 \right] \eta, \tag{3.2}$$

with α the infinitesimal parameter of a $U(1)$ gauge transformation. Adding it all together we can write

$$\mathcal{L}_{\rm tot} = \mathcal{L} + \mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP} = \mathcal{L}_{\rm cl}(\phi) + \mathcal{L}_{\rm free} + \mathcal{L}_{\rm int}(t). \tag{3.3}$$

The purely classical terms are in $\mathcal{L}_{\rm cl}$. The free Lagrangian contains the time-independent terms quadratic in the fluctuation fields, from which the free propagators are constructed. The interaction Lagrangian contains all other terms, which are treated as perturbations. Explicitly,

$$\begin{aligned}
\mathcal{L}_{\rm cl} &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) && (3.4)\\
\mathcal{L}_{\rm free} &= -\frac{1}{2} A^\mu \left[-g_{\mu\nu} (\partial^2 + g^2 \phi_0^2) + \partial_\mu \partial_\nu (1 - \frac{1}{\xi}) \right] A^\nu - \bar{\eta} \left[\partial^2 + \xi g^2 \phi_0^2 \right] \eta \\
&\quad - \frac{1}{2} h \left[\partial^2 + V_{hh}(0) \right] h - \frac{1}{2} \theta \left[\partial^2 + V_{\theta\theta}(0) + \xi g^2 \phi_0^2 \right] \theta && (3.5)\\
\mathcal{L}_{\rm int} &= -h \left[\partial^2 \phi + V_\phi \right] + \frac{g^2}{2} (\phi^2 - \phi_0^2) \left[A_\mu A^\mu - \xi \theta^2 - 2\xi \bar{\eta} \eta \right] \\
&\quad - 2g \partial_\mu \phi A^\mu \theta - \frac{1}{2} (V_{hh}(t) - V_{hh}(0)) h^2 - \frac{1}{2} (V_{\theta\theta}(t) - V_{\theta\theta}(0)) \theta^2 + \ldots, && (3.6)
\end{aligned}$$

with $\phi_0 = \phi(0)$ the initial field value. The ellipses denote terms of third or higher order in the fluctuation fields. As before $V_\phi = \partial_\phi V$ etc. with V the classical potential.

We define the "mass"-matrix via

$$m^2_{\alpha\beta} = -\frac{\partial^2 \mathcal{L}}{\partial \chi_\alpha \partial \chi_\beta} = \bar{m}^2_{\alpha\beta} + \delta m^2_{\alpha\beta}(t), \qquad \chi_\alpha = \{A^\mu, \eta, h, \theta\} \tag{3.7}$$

which can be split in a free time-independent part, denoted by an overbar, and a time-dependent part. The non-zero elements of the mass matrix are:

$$\begin{aligned}
&m^2_{A^\mu A^\nu} = -g^2 \phi^2 g^{\mu\nu} \equiv -m_A^2 g^{\mu\nu}, \quad m^2_\eta = \xi g^2 \phi^2, \quad m_h^2 = V_{hh}, \quad m_\theta^2 = V_{\theta\theta} + \xi g^2 \phi^2, \\
&m^2_{\theta A^\mu} = 2g\dot{\phi} \delta^\mu_0 \equiv m^2_{A\theta} \delta^\mu_0, && (3.8)
\end{aligned}$$

where for the diagonal entries we used the notation $m^2_\alpha = m^2_{\alpha\beta} \delta^\alpha_\beta$. The only off-diagonal term is the term in the second line above mixing the Goldstone boson and the temporal part of the gauge field. The temporal gauge boson has a wrong sign mass. As it also has a wrong sign kinetic term, the dispersion relation for A^0 is still of the standard form $\omega^2_{A^0} = \vec{k}^2 + |m^2_{A^0 A^0}| = \vec{k}^2 + m_A^2$.

From the interaction Lagrangian we can read off the n-point functions. Of particular interest for the calculations performed in this paper are the following one-point, two-point and three-point vertices:

$$\Gamma_h = -i(\Box \phi + V_\phi), \qquad \Gamma_{\alpha\beta} = -i m^2_{\alpha\beta}, \qquad \Gamma_{h\alpha\beta} = -i \partial_\phi m^2_{\alpha\beta}. \tag{3.9}$$

Figure 1: The classical background field equation can be derived from the tadpole diagram with one external h^+ leg as shown in the figure. The cross represents the one-point function.

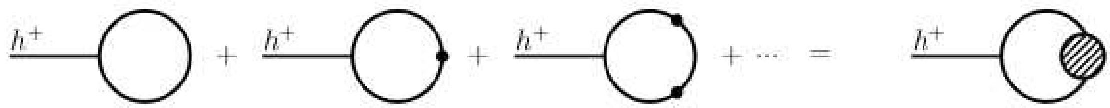

Figure 2: The background field equation of motion is corrected by all one-loop diagrams with one external h^+ leg as shown in the figure. The lines correspond to the bare propagator, and the dots to two-point (mass) insertions. This can be resummed to get a resummed propagator running in the loop, depicted here by a blob.

3.2 Real scalar field

To warm up, we first perform the one-loop calculation for a single real scalar field rolling down the potential, using the Schwinger-Keldysh formalism. We wish to calculate the one-loop correction to the equation of motion, defined as the sum over all one-loop diagrams with one external leg (of the quantum field h) shown in Figure (2). Integrating the equations of motion, we retrieve the standard Coleman-Weinberg potential (2.12) in the time-independent limit $\dot{\phi}(t) = 0$. We follow the treatment in [36, 37].

Consider a real scalar field expanded around a classical field value $\Phi = \phi(t) + h(x,t)$, where we can split $\phi(t) = \phi_0 + \delta\phi(t)$ with $\delta\phi(0) = 0$. The one-loop correction to the equation of motion comes from the sum of all vacuum loops with one external leg, as depicted in Figure (2), which can be expressed as [31, 32, 41, 42]

$$\begin{aligned} 0 &= \frac{\delta\Gamma}{\delta\phi_+}\bigg|_{\{J_+=J_-=0\}} = i\left(\Gamma_h + \frac{1}{2}\Gamma^+_{hhh}G_h^{++}(0)\right)\bigg|_{\{J_+=J_-=0\}} + \mathcal{O}(h^2) \\ &= \Box\phi + V_\phi + \frac{1}{2}(\partial_\phi m^2_{hh})G_h^{++}(0) + \mathcal{O}(h^2), \end{aligned} \tag{3.10}$$

where in the second line we have set $\phi_+|_{\{J_\pm=0\}} = \phi$. $G_h^{++}(x,x')$ is the dressed or resummed propagator taking all two-point insertions into account, which is defined in appendix A.2. The first term is the classical tree level contribution to the equation of motion, which can be found from the tadpole diagram in Figure (1). The second term is the 1-loop correction, which is the sum of all one-loop diagrams with one external h^+ leg and arbitrary number of mass insertions, as depicted in Figure (2). The factor 1/2 is a symmetry factor originating from the reflection symmetry of the Feynman diagrams. All the relevant n-point functions are defined in (3.9).

We only need to consider the 1-loop contribution on the (+)-branch of the Schwinger-Keldysh in-in formalism (the calculation on the (−)-branch gives the same result). Therefore the calculation is fully analogous to the usual in-out scattering matrix calculation. For ease of notation we drop the (++) superscript in the following.

The dressed propagator can be expressed in terms of the mode functions

$$G_h(0) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{|U_h|^2}{2\bar{\omega}_h}, \tag{3.11}$$

where for notational convenience we dropped the subscript $\vec{k}$ on the mode functions and the frequency. The mode functions satisfy a wave equation with a time-dependent frequency (which can be read off from the quadratic part of the Lagrangian — see appendix A for more details)

$$\left[\partial_t^2 + \omega_{\vec{k},h}^2\right] U_h(t) = 0, \qquad \text{with } \; U_h(0) = 1, \;\; \dot{U}_h(0) = -i\bar{\omega}_h. \tag{3.12}$$

The frequency can be split in a time-independent and a time-dependent piece $\omega_h^2(t) = \bar{\omega}_h^2 + \delta m_h^2(t)$ with $\bar{\omega}_h^2 = \vec{k}^2 + \bar{m}_h^2$, with as before the overbar denoting the time-independent quantities. To solve the mode equation (3.12) we make the Ansatz

$$U_h = \mathrm{e}^{-i\bar{\omega}_h t}(1 + f_h(t)). \tag{3.13}$$

The function f_h satisfies $\ddot{f}_h - 2i\bar{\omega}_h \dot{f}_h = -\delta m_h^2(1 + f_h)$ and has boundary conditions $\dot{f}_h(0) = f_h(0) = 0$. This can be solved using the Green's function method to yield:

$$f_h = -\frac{1}{\bar{\omega}_h} \int_0^t \mathrm{d}t' \sin(\bar{\omega}_h \Delta t) \mathrm{e}^{i\bar{\omega}_h \Delta t}(1 + f_h(t'))\delta m_h^2(t'), \tag{3.14}$$

with $\Delta t = t - t'$. We can solve the mode equations iteratively order by order in mass insertions $f = f^{(1)} + f^{(2)} + ...$ To isolate the divergent part it is enough to only go to first order, since $|U_h|^2 = 1 + 2\mathrm{Re} f_h^{(1)} + \mathcal{O}(k^{-4})$ — since, as we will see in a moment, for large momentum $f_h^{(1)} \propto k^{-2}$. For $f_h(t) = 0$ we get back the bare (free) propagator with no mass insertion.[2] Define $f_h^{(1)}$ as the first order correction in the mass insertion; it is given by

$$f_h^{(1)} = -\frac{1}{\bar{\omega}_h} \int_0^t \mathrm{d}t' \sin(\bar{\omega}_h \Delta t) \mathrm{e}^{i\bar{\omega}_h \Delta t} \delta m_h^2(t'). \tag{3.15}$$

Using partial integration, and taking the real part gives

$$\mathrm{Re} f_h^{(1)} = -\frac{\delta m_h^2(t)}{4\bar{\omega}_h^2} + \frac{1}{4\bar{\omega}_h^2} \int \mathrm{d}t' \cos(2\bar{\omega}_h \Delta t) \partial_{t'}(\delta m_h^2(t')) = -\frac{\delta m_h^2(t)}{4\bar{\omega}_h^2} + \mathcal{O}(\bar{\omega}_h^{-3}). \tag{3.16}$$

Finally

$$\begin{aligned} G_h(0) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1 + 2\mathrm{Re} f_h^{(1)} + ...}{2\bar{\omega}_h} = \frac{1}{4\pi^2} \int k^2 \mathrm{d}k \left(\frac{1}{k} - \frac{1}{2k^3}(\bar{m}_h^2 + \delta m_h^2) + \mathcal{O}(k^{-5}) \right) \\ &= \frac{1}{8\pi^2} \left(\Lambda^2 - \frac{1}{2} m_h^2(t) \ln \left(\frac{\Lambda^2}{m_h^2} \right) \right) + \text{finite}. \end{aligned} \tag{3.17}$$

[2]Note that $f_h(t) = 0$ corresponds to the first order result in the perturbative calculation of appendix B, while $f_h^{(1)}$ corresponds to the second order result in the perturbative calculation.

The 1-loop equation of motion (3.10) thus becomes

$$0 = \Box\phi + V_\phi + \frac{\partial_\phi m_h^2}{16\pi^2}\left(\Lambda^2 - \frac{1}{2}m_h^2(t)\ln\left(\frac{\Lambda^2}{m_h^2}\right)\right). \tag{3.18}$$

We can integrate the last term with respect to ϕ to get the one-loop correction to the effective potential. Up to field-independent and finite terms:

$$\Gamma^{1-\text{loop}} = -\int \mathrm{d}^4x \int \mathrm{d}\phi \frac{\partial_\phi m_h^2}{16\pi^2}\left(\Lambda^2 - \frac{1}{2}m_h^2\ln\left(\frac{\Lambda^2}{m_h^2}\right)\right) = -\frac{1}{16\pi^2}\int \mathrm{d}^4x \left(m_h^2\Lambda^2 - \frac{1}{4}m_h^4\ln\left(\frac{\Lambda^2}{m_h^2}\right)\right). \tag{3.19}$$

In the static limit that the background field, and thus the mass term, is time-independent $\Gamma^{1-\text{loop}} = -\int \mathrm{d}^4x V_{CW}$, and we recover the Coleman-Weinberg result (2.12).

3.3 Abelian Higgs model

We now extend the analysis to a U(1) model with a complex Higgs field. The one-loop equation of motion (3.10) generalizes to

$$0 = \Box\phi + V_\phi + \frac{1}{2}\left(\partial_\phi m_{\alpha\beta}^2\right) G_{\alpha\beta}^{++}(0). \tag{3.20}$$

We use the Feynman-'t Hooft gauge $\xi = 1$, for which the equations of motion of A^i and A^0 decouple. All four components of the gauge field satisfy a Klein-Gordon equation. The quadratic terms for $\alpha = \{h, \eta, A^i\}$ are diagonal, and the one-loop calculation proceeds analogously to the scalar field case discussed in the previous subsection. On the other hand, the fields $\{A^0, \theta\}$ couple in the quadratic terms, because of the non-diagonal mass term $m_{A^0\theta}^2 \neq 0$, and need to be treated with care.

The calculation of the propagator for the real scalar h was done in the previous subsection. Also for η, which is an anti-commuting complex scalar, the scalar field result applies with a factor 2 for the 2 real d.o.f. and a minus sign to take into account the anti-commuting nature. In the $\xi = 1$ gauge the propagator G_{A^i} satisfies $[\Box + m_A^2]G_{A^i} = -i\delta(x - x')$. As this equation is of the same form as the one for the scalar field propagator, the scalar field results can be applied. Hence, A^i contributes as three scalars with mass m_A each. The result thus is

$$\begin{aligned}\sum_{\{h,\eta,A^i\}} \frac{1}{2}(\partial_\phi m_\alpha^2)G_\alpha^{++}(0) = \frac{1}{16\pi^2}\Bigg[& \quad \left(\partial_\phi m_h^2\right)\left(\Lambda^2 - \frac{1}{2}m_h^2\ln\left(\frac{\Lambda^2}{m_h^2}\right)\right) \\ & -2 \quad \left(\partial_\phi m_\eta^2\right)\left(\Lambda^2 - \frac{1}{2}m_\eta^2\ln\left(\frac{\Lambda^2}{m_\eta^2}\right)\right) \\ & +3 \quad \left(\partial_\phi m_{A^i}^2\right)\left(\Lambda^2 - \frac{1}{2}m_{A^i}^2\ln\left(\frac{\Lambda^2}{m_{A^i}^2}\right)\right)\Bigg].\end{aligned} \tag{3.21}$$

The difficulty is in calculating the propagators for $\{A^0, \theta\}$, as these fields couple in their equations of motion. We only outline the calculation, more details can be found in [36, 37].

We define two sets $\alpha = \{1,2\}$ of mode functions which satisfy (following from the quadratic part of the Lagrangian, see appendix A)

$$\left[\begin{pmatrix} -\left(\partial_t^2 + \bar{\omega}_A^2\right) & 0 \\ 0 & \partial_t^2 + \bar{\omega}_\theta^2 \end{pmatrix} + \begin{pmatrix} -\delta m_A^2 & \delta m_{A\theta}^2 \\ \delta m_{A\theta}^2 & \delta m_\theta^2 \end{pmatrix}\right]\begin{pmatrix} U_A^\alpha \\ U_\theta^\alpha \end{pmatrix} = 0, \tag{3.22}$$

with

$$U_m^\alpha(0) = \delta_m^\alpha, \qquad \dot{U}_m^\alpha(0) = -i\bar{\omega}_m \delta_m^\alpha. \tag{3.23}$$

δm_m^2 and δm_{mn}^2 correspond to the diagonal and off-diagonal entries of the time-dependent part of the mass matrix. For example: $\bar{m}_A^2 = g^2\phi_0^2$, $\delta m_A^2 = g^2\left(\phi^2 - \phi_0^2\right)$. The frequency for the temporal gauge field is $\omega_A^2 = k^2 + m_A^2$. The $\alpha = 1$ mode is the "mostly gauge boson" mode, and $\alpha = 2$ is the "mostly Goldstone boson mode". The modes do not decouple because of the off-diagonal δm_{mn}^2 term. The resummed equal-time propagator in terms of the mode functions is

$$G_{kn}(0) = \int \frac{d^3k}{(2\pi)^3}\left[-\frac{1}{4\bar{\omega}_A}\left(U_k^1 U_n^{1*} + U_k^{1*} U_n^1\right) + \frac{1}{4\bar{\omega}_\theta}\left(U_k^2 U_n^{2*} + U_k^{2*} U_n^2\right)\right] \tag{3.24}$$

and thus

$$\begin{aligned}\sum_{\{\theta,A^0\}} \frac{1}{2}\left(\partial_\phi m_{\alpha\beta}^2\right) G_{\alpha\beta}^{++} = \frac{1}{2} \ &\int \frac{\mathrm{d}^3k}{(2\pi)^3}\Bigg[\partial_\phi m_A^2\left(\frac{1}{2\bar{\omega}_A}|U_A^1|^2 - \frac{1}{2\bar{\omega}_\theta}|U_A^2|^2\right) \\ &+ \ \partial_\phi m_\theta^2\left(\frac{1}{2\bar{\omega}_\theta}|U_\theta^2|^2 - \frac{1}{2\bar{\omega}_A}|U_\theta^1|^2\right) \\ &+ \ 2\partial_\phi m_{\theta A}^2\left(-\frac{1}{4\bar{\omega}_A}(U_A^1 U_\theta^{1*} + U_A^{1*} U_\theta^1) + \frac{1}{4\bar{\omega}_\theta}(U_A^2 U_\theta^{2*} + U_A^{2*} U_\theta^2\right)\Bigg].\end{aligned} \tag{3.25}$$

To solve for the mode functions make the Ansatz which is consistent with the boundary conditions if we again choose $f(0) = \dot{f}(0) = 0$:

$$\begin{aligned} U_A^1 &= \mathrm{e}^{-i\bar{\omega}_A t}(1 + f_A^1), \qquad U_\theta^1 = \mathrm{e}^{-i\bar{\omega}_\theta t} f_\theta^1, \\ U_\theta^2 &= \mathrm{e}^{-i\bar{\omega}_\theta t}(1 + f_\theta^2), \qquad U_A^2 = \mathrm{e}^{-i\bar{\omega}_A t} f_A^2. \end{aligned} \tag{3.26}$$

We can again solve iteratively, and define an expansion in terms of mass-term insertions $f_m^\alpha = f_m^{\alpha(1)} + f_m^{\alpha(2)}$.... To isolate the divergent part of the one-loop potential we again only need the first order result. Plugging the Ansatz (3.26) in the mode equations gives

$$\begin{aligned} \ddot{f}_\alpha^{m(1)} - 2i\bar{\omega}_\alpha \dot{f}_\alpha^{m(1)} &= -\delta m_\alpha^2, && \text{for } \{m,\alpha\} = \{1,A\},\ \{2,\theta\} \\ \ddot{f}_\alpha^{m(1)} - 2i\bar{\omega}_\alpha \dot{f}_\alpha^{m(1)} &= (-1)^m \delta m_{A\theta}^2 \mathrm{e}^{(-1)^m i(\bar{\omega}_A - \bar{\omega}_\theta)t}, && \text{for } \{m,\alpha\} = \{1,\theta\},\ \{2,A\} \end{aligned} \tag{3.27}$$

where we only kept the highest order results. To do so we used that at large momentum $\omega^n \partial_t^m f^{(l)} \propto k^{m+n-2l}$. Just as in the scalar field case, the equations can be solved using the

Green's function method. The f^1_A and f^2_θ equations are exactly the same as found for the scalar in the previous subsection, and hence give the same result:

$$
\begin{aligned}
f^{m(1)}_\alpha &= -\frac{1}{\bar\omega_\alpha}\int^t \mathrm{d}t' \sin(\bar\omega_\alpha \Delta t) \mathrm{e}^{i\bar\omega_\alpha \Delta t}\delta m^2_\alpha(t'), && \text{for } \{m,\alpha\} = \{1,A\},\ \{2,\theta\},\\
f^{m(1)}_\alpha &= \frac{(-1)^m}{\bar\omega_\alpha}\int^t \mathrm{d}t' \sin(\bar\omega_\alpha \Delta t) \mathrm{e}^{i\bar\omega_\alpha \Delta t}\mathrm{e}^{(-1)^m i(\bar\omega_A-\bar\omega_\theta)t'}\delta m^2_{A\theta}(t'), && \text{for } \{m,\alpha\} = \{1,\theta\},\ \{2,A\}.
\end{aligned}
\tag{3.28}
$$

Now consider the first line of (3.25). The terms $|U^2_A|^2 = |f^{2(1)}_A|^2$ and $|U^1_\theta|^2 = |f^{1(1)}_\theta|^2$ are second order in f and thus give no contribution to the divergent terms. The remaining terms on this line are analogous to the scalar loop, they correspond to Feynman diagrams with θ and A^0 loop running in the loop, and give the standard Coleman-Weinberg result. Hence we get a contribution as in (3.17) but now for θ, A^0. Remains to evaluate the second line of (3.25):

$$
\begin{aligned}
&\partial_\phi m^2_{A\theta}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(-\frac{1}{4\bar\omega_A}2\mathrm{Re}[\mathrm{e}^{it(\bar\omega_A-\bar\omega_\theta)}f^{1(1)}_\theta] + (\{1,A\}\leftrightarrow\{2,\theta\})\right)\\
&= \frac{\partial_\phi m^2_{A\theta}}{4}\int\frac{d^3k}{(2\pi)^3}\left[\frac{\cos[(\bar\omega_A-\bar\omega_\theta)t]}{\bar\omega_A\bar\omega_\theta}\int_0^{t'} dt' \sin[2\bar\omega_\theta\Delta t]\cos[(\bar\omega_\theta-\bar\omega_A)t']\delta m^2_{A\theta}(t') + (a\leftrightarrow\theta)\right]\\
&= \frac{\partial_\phi m^2_{A\theta}}{8}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\bar\omega_A\bar\omega_\theta(\bar\omega_A+\bar\omega_\theta)}\cos^2[(\bar\omega_A-\bar\omega_\theta)t]\delta m^2_{A\theta}(t) + \mathcal{O}(\omega^{-4})\\
&= \frac{\partial_\phi m^2_{A\theta}}{4\pi^2}m^2_{A\theta}\ln\left(\frac{\Lambda^2}{m^2_{A\theta}}\right) + \text{finite}.
\end{aligned}
\tag{3.29}
$$

To obtain the third line we used partial integration. Further $m^2_{A\theta} = \delta m^2_{A\theta}$, as there is no time-independent mix term.

Adding it all up, the one-loop equation of motion thus becomes

$$
0 = \Box\phi + V_\phi + \frac{\partial_\phi m^2_{A\theta}}{4\pi^2}m^2_{A\theta}\ln\left(\frac{\Lambda^2}{m^2_{A\theta}}\right) + \sum_{\{\phi,\eta,A^i,\theta,A_0\}}\frac{S_i}{16\pi^2}\partial_\phi m_i^2\left(\Lambda^2 - \frac{1}{2}m_i^2\ln\left(\frac{\Lambda^2}{m_i^2}\right)\right) \tag{3.30}
$$

with $S_i = \{1,-2,3,1,1\}$ for $i = \{\phi,\eta,A^i,\theta,A_0\}$ counting the degrees of freedom. Since the equations of motion follow from the functional derivative of the effective action, one can invert the process, and find (up to a field-independent constant) the effective action by integrating the field equations with respect to ϕ:

$$
\begin{aligned}
\Gamma^{1-\mathrm{loop}} &= -\frac{1}{16\pi^2}\int\mathrm{d}^4x\Big[\Lambda^2\left(m_h^2 - 2m_\eta^2 + 3m^2_{A^i} + m^2_\theta + m^2_{A_0}\right) \\
&\qquad -\frac{1}{4}\ln\Lambda^2\left(m_h^4 - 2m_\eta^4 + 3m^4_{A^i} + m^4_\theta + m^4_{A_0} - 2m^4_{\theta A}\right)\Big] + \text{finite}\\
&= -\frac{1}{16\pi^2}\int\mathrm{d}^4x\left[\Lambda^2\left(V_{hh} + V_{\theta\theta} + 3m_A^2\right) - \frac{1}{4}\ln\Lambda^2\left(V_{hh}^2 + V_{\theta\theta}^2 + 3m_A^4 - 6V_{\theta\theta}m_A^2\right)\right].
\end{aligned}
\tag{3.31}
$$

In the second step we used the zeroth order background equation of motion and gauge invariance to write

$$\int \mathrm{d}t\, m_{\theta A}^4 = 4g^2 \int \mathrm{d}t\dot{\phi}^2 = -4g^2 \int \mathrm{d}t\phi\ddot{\phi} = 4g^2 \int \mathrm{d}t\phi V_\phi = 4 \int \mathrm{d}t\, m_A^2 V_{\theta\theta} \tag{3.32}$$

up to higher loop corrections. With this substitution the final expression is in terms of explicitly gauge independent quantities. This can be seen more explicitly in the perturbative calculation in appendix A, which is done for arbitrary gauge parameter ξ. As a result the *on-shell* one loop effective potential is gauge invariant. In the static limit $V_{\theta\theta} \to 0$ and all other masses are time-independent, our results reproduce the standard Coleman-Weinberg potential (2.12).

The gauge independent part of the Goldstone boson mass $V_{\theta\theta}$ appears explicitly in the one-loop potential. Except for the very last term in (3.31), the one loop potential can be obtained from the Coleman-Weinberg potential, treating θ as a physical bosonic degree of freedom. The calculation done in unitary gauge with θ completely "gauged away" from the potential (which, as discussed in subsection 2.2, for ϕ displaced away from its minimum seems only possible on-shell) gives the wrong answer. This answers the question posed at the beginning of this section. The Goldstone boson cannot be removed "by hand", and keeping its contribution in the one-loop potential assures this is continuous.

Our answers disagree with the naive expectation obtained in unitary gauge, where the Goldstone boson is absent. The reason is that unitary gauge is a singular limit. It corresponds to taking the limit $\xi \to \infty$ such that the θ propagator vanishes. This procedure, however, does not commute with the $k \to \infty$ limit taken in the momentum integrals to isolate the divergent terms. That unitary gauge gives an incorrect result has been noted before [24]. In this gauge higher order loop corrections affect the leading term and must be taken into account [48].

The last term on the last line of (3.31) can be interpreted as a correction to the Coleman-Weinberg potential, due the fact that ϕ is rolling down its potential rather than sitting in its minimum. It vanishes in the static limit; note in this respect that it came from the $\dot{\phi}$ term.

3.4 Fermions

Even if the focus in this article is obviously on scalar fields, we want to include a section on fermionic fields here, in order to arrive at a more complete picture of one-loop corrections in a theory with a displaced Higgs field. In Standard Model Higgs inflation the top quark contributes significantly to the one-loop potential, whereas in supersymmetric theories Higgsinos and gauginos should be taken into account as well. The full calculation for fermions has been done in [38]. Here we summarize their results, adapted to calculate the effective potential.

In a supersymmetric theory, the gauginos and Higgsinos couple in the mass matrix if the gauge symmetry is broken. It is always possible to diagonalize the mass matrix, and do the calculation in terms of mass eigenstates, whether the theory is supersymmetric or not. There are no mixed loops, such as in the bosonic sector, where the Goldstone boson and temporal

gauge field are coupled. In the static limit, the one-loop is given by the Coleman-Weinberg potential (2.12), to which each mass eigenstate contributes. To find possible time-dependent corrections, one can again use the CTP formalism. We calculate the one-loop correction to the equation of motion for the background field due to a fermion loop. Only fermions which have a field dependent mass term contribute.

Consider a Dirac or Majorana fermion with Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi(t))\psi. \tag{3.33}$$

For a Yukawa type interaction the fermion mass is $m_F = \lambda\phi$, with λ the Yukawa coupling and ϕ the Higgs field. The one-loop equation of motion for the background field is [38]

$$0 = \Box\phi + V' + \frac{1}{2}\sum_{\text{bosons}} (\partial_\phi m_{\alpha\beta})\, G^{++}_{\alpha\beta}(x,x) - \frac{1}{2}\sum_{\text{fermions}} (\partial_\phi m_{\alpha\beta})\, G^{++}_{\alpha\beta}(x,x). \tag{3.34}$$

The bosonic contribution was calculated in the previous section, here we concentrate on the second fermionic contribution. For simplicity we do the calculation for a single fermion field. The equal-time dressed propagator for a fermion is given in the appendix (A.20). The Dirac equation can be rewritten as a second order wave equation, using a particular Ansatz for the spinors (A.17, A.18). This maps the problem to an equivalent form as for the real scalar discussed in section 3.2. The one-loop potential can be calculated analogously. The result found in [38] is

$$-\sum_{\text{d.o.f.}} \frac{1}{2}(\partial_\phi m_\psi)\, G^{++}_\psi(0) = -\sum_{\text{d.o.f.}} \frac{\partial_\phi m_\psi}{8\pi^2}\left[m_\psi\Lambda^2 - \frac{1}{2}\left(m_\psi^3 + \frac{1}{2}\ddot{m}_\psi\right)\ln\left(\frac{\Lambda^2}{m_\psi^2}\right)\right], \tag{3.35}$$

where the sum is over all helicity states, 4 for a Dirac fermion and 2 for a Majorana/Weyl fermion. For a Yukawa mass we have on-shell

$$\frac{\ddot{m}_\psi}{m_\psi} = \frac{\ddot{\phi}}{\phi} = -V_{\theta\theta}. \tag{3.36}$$

Integrating the field equations to get the the one-loop correction to the effective action gives

$$\Gamma^{1-\text{loop}} = \sum_{\text{d.o.f.}} \frac{1}{16\pi^2}\int \mathrm{d}^4x\left[m_\psi^2\Lambda^2 - \frac{1}{4}(m_\psi^4 - m_\psi^2 V_{\theta\theta})\ln\Lambda^2\right] + \text{finite}. \tag{3.37}$$

In the static limit $V_{\theta\theta} \to 0$, this indeed reproduces the standard CW result (2.12). We thus find that the time-dependent corrections scale with the Goldstone boson mass.

4. Conclusions and outlook

In this work we have computed the one-loop corrected equations of motion for the background Higgs field, and, by integrating, the one-loop effective action for a theory in which the Higgs field is slowly rolling down its potential. For our U(1) toy model with a complex Higgs field

$\Phi = \phi_0 + \delta\phi(t) + h(x,t) + i\theta(x,t)$ moving through a potential V and a vector field $A^\mu(x,t)$ we find

$$\Gamma = \int \mathrm{d}^4x \left\{ \mathcal{L}_{\rm cl} - \frac{1}{16\pi^2}\left[\Lambda^2\left(V_{hh} + V_{\theta\theta} + 3m_A^2\right) - \frac{1}{4}\ln\Lambda^2\left(V_{hh}^2 + V_{\theta\theta}^2 + 3m_A^4 - 6V_{\theta\theta}m_A^2\right)\right]\right\} + \mathcal{O}(\hbar^2). \tag{4.1}$$

up to finite and field-independent terms. To write the results in this manifestly gauge invariant way we used the zeroth order background equations of motion and gauge invariance to replace $m_{A\theta}^4 \to m_A^2 V_{\theta\theta}$ in the one-loop correction. The potential is completely arbitrary. We first remark that in the static case one has $V_{\theta\theta} = 0$ and we are left with the well-known Coleman-Weinberg result. Note that the last term in (4.1) can change the sign of the log term, but only if all masses are of the same order. If the scalar and gauge boson masses are hierarchical, it will be negligible. This may be important for Higgs inflation in certain GUT models.

With the Higgs field displaced from its minimum, the Goldstone boson θ is massive. It cannot be removed from the theory. At the classical level we can still use unitary gauge (and the equation of motion) to eliminate the Goldstone boson from the theory, at the quantum level this procedure gives wrong results. In particular, the Goldstone boson still contributes to the effective action as if it was a massive scalar degree of freedom. This comes in addition to the contribution from the massive gauge boson. Thus even if we should not call the Goldstone boson "physical" (its associated degree of freedom, after all, has been used to give the gauge boson a mass), the factors of $V_{\theta\theta}$ in the potential are real and can not be discarded. (One might argue that they are induced by the massive gauge boson.)

The equivalent calculation performed in unitary gauge gives wrong answers. The reason is that unitary gauge is ill-defined. It corresponds to taking the limit $\xi \to \infty$ such that the θ propagator vanishes. This procedure, however, does not commute with the $k \to \infty$ limit taken in the momentum integrals to isolate the divergent terms. Problems with unitary gauge were noted before, for example in the calculation of the one-loop potential at finite temperature [24]. In that context it was shown that two-loop effects contribute at the same order, and cannot be neglected [48].

Our results imply that supersymmetric Higgs inflation is free of quadratic divergencies, as the bosonic and fermionic degrees of freedom still cancel. In addition the effective potential is continuous in going from the symmetric to the broken phase, as it should be.

Our calculations closely followed the work of Heitmann and Baacke, generalized to an arbitrary potential. Moreover we explicitly show that the results are gauge invariant on shell. Our results reproduce the Coleman-Weinberg results in the static limit. Ref. [43] has calculated the effective potential in terms of manifestly gauge invariant quantities, but only in the adiabatic limit, which does not take into account the time-dependence of the rolling Higgs field. These time-dependent corrections are essential for us to show the gauge-independence of the final result.

To get from our toy model to the case of Higgs inflation the first step is to generalize the gauge group U(1) to the Standard Model or GUT gauge group, depending on the inflation

model under consideration. This is a trivial extension of our results. The second, far less trivial, step is to do the calculation in a Friedmann-Robertson-Walker spacetime rather than in Minkowski spacetime. The scalar and fermion field contributions can rather straightforwardly be generalized, and yield additional corrections to the Coleman-Weinberg potential due to the expansion of the universe. But the difficulties arise in the gauge boson and Goldstone boson sector. In a cosmological spacetime Lorentz symmetry is broken, and as a consequence the temporal and longitudinal/transversal parts of the gauge field no longer decouple. This is left for future work.

A third step left to be done is generalizing the results to non-canonical kinetic terms. If the kinetic terms cannot be diagonalized by simple field redefinitions, as is the case in Standard Model Higgs inflation, the radial Higgs field and Goldstone bosons couple in a non-trivial way. The equations can still be solved in the adiabatic approximation. However, different approximation schemes have to be developed if the field evolution is fast, which is the case after inflation.

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A. CTP formalism

In the usual S-matrix approach, also called in-out formalism, the generating functional describes the transition from an in-state vacuum in the past to an out-state vacuum in the future $Z[J] = \langle 0, t_{\rm in} | 0, t_{\rm out} \rangle_J$, which is calculated in the presence of an external source J. In the path-integral formulation

$$Z[J] = \int \mathcal{D}\phi \, \mathrm{e}^{iS[\phi] + \int \mathrm{d}^4 x J\phi}. \tag{A.1}$$

This formalism is well suited to calculate scattering amplitudes, processes in which the out-state is known. In non-equilibrium situations it is more useful to calculate the physically relevant field expectation values of an observable $\langle 0, t_{\rm in} | \mathcal{O} | 0, t_{\rm in} \rangle$ taken with respect to the same states. The generating functional in this in-in formalism, also known as Schwinger-Keldysh or closed time-path (CTP) formalism [26, 27, 28, 29, 30, 31, 32, 33], is defined employing two external sources:

$$Z[J^+, J^-] = {}_{J_-}\langle 0, t_{\rm in} | 0, t_{\rm in} \rangle_{J_+} = \sum_\alpha \langle 0, t_{\rm in} | \alpha, t_{\rm out} \rangle_{J_-} \langle \alpha, t_{\rm out} | 0, t_{\rm in} \rangle_{J_+}, \tag{A.2}$$

where the sum goes over a complete set of out states. The above expression can be understood as the in-vacuum going forward in time under influence of the J_+ source, and then returning

back in time under the influence of the J_- source. On both branches propagators and vertices can be defined, with the $-$-branch giving the time reversed of expressions the $+$-branch.

A.1 Free propagators

We will define free propagators and vertices, needed for the one-loop perturbative calculation. The free Lagrangian (3.5) is of the form $\mathcal{L}^{\rm free} = -(1/2)\sum_i \chi_i(x^\mu)\bar{K}^i(x^\mu)\chi_i(x^\mu)$, with the sum over all (bosonic) fields $\chi_i = \{h, \theta, \eta, A^\mu\}$. The time-dependent parts of the quadratic action are treated as interactions. As before, the overbar denotes that we only consider the time-independent parts of the quadratic terms. The free propagators are defined as

$$\begin{pmatrix} \bar{K}^i(x^\mu) & 0 \\ 0 & -\bar{K}^i(x^\mu) \end{pmatrix} \begin{pmatrix} \bar{G}_i^{++}(x^\mu - y^\mu) & \bar{G}_i^{+-}(x^\mu - y^\mu) \\ \bar{G}_i^{-+}(x^\mu - y^\mu) & \bar{G}_i^{--}(x^\mu - y^\mu) \end{pmatrix} = -i\delta(x^\mu - y^\mu)\mathbf{I}_2. \tag{A.3}$$

These equations can be easily solved in Fourier space, for example the $(++)$ Green's function is

$$\begin{aligned} \bar{G}_i^{++}(k) &= \frac{i}{k^2 - \bar{m}_i^2 + i\epsilon} \\ (\bar{G}_A^{++})_{\mu\nu}(k) &= -\frac{i}{k^2 - \bar{m}_A^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{i\xi}{k^2 - \bar{m}_\xi^2 + i\epsilon}\left(\frac{k_\mu k_\nu}{k^2}\right), \end{aligned} \tag{A.4}$$

where the first expression applies to the scalars $i = \{h, \theta, \eta\}$, and the second to the vector boson. Here the masses correspond to the *time-independent* parts of the mass terms (3.8), indicated by the overbar, appearing in $\mathcal{L}^{\rm free}$. Explicitly

$$\bar{m}_A^2 = g^2\phi_0^2, \qquad \bar{m}_\eta^2 = \bar{m}_\xi^2 = \xi g^2\phi_0^2, \qquad \bar{m}_h^2 = V_{hh}, \qquad \bar{m}_\theta^2 = V_{\theta\theta} + \bar{m}_\xi^2. \tag{A.5}$$

The time-independent frequencies are defined as before $\bar{\omega}_i^2 = k^2 + \bar{m}_i^2$. In real space

$$\begin{aligned} \bar{G}_i^{++}(x^\mu - y^\mu) &= \langle 0|T(\chi^i(x^\mu)\chi^i(y^\mu))|0\rangle = \int \frac{{\rm d}^4k}{(2\pi)^4}{\rm e}^{-ik^\mu(x-y)_\mu}\bar{G}_i^{++}(k) \\ &= \int \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{2\bar{\omega}_i}{\rm e}^{-ik^\mu(x-y)_\mu}\Theta(x^0 - y^0) + \int \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{2\bar{\omega}_i}{\rm e}^{ik^\mu(x-y)_\mu}\Theta(y^0 - x^0) \\ &= \bar{G}_i^{-+}(x^\mu - y^\mu)\Theta(x^0 - y^0) + \bar{G}_i^{+-}(x^\mu - y^\mu)\Theta(y^0 - x^0) \end{aligned} \tag{A.6}$$

and $\bar{G}_i^{--}(x^\mu - y^\mu) = \bar{G}_i^{++}(y^\mu - x^\mu)$. In the second step we performed the contour integral over k^0. A similar derivation can be done for the gauge boson propagators. In the one-loop calculation we only need certain contracted expressions. These can be expressed in terms of the scalar propagator above (A.6), with now $i = \{A, \xi\}$ (the equations apply equally well to all $(\pm\pm)$-Green's functions).

$$g^{\mu\nu}\bar{G}_{A^\mu A^\nu} = -3\bar{G}_A - \xi\bar{G}_\xi \tag{A.7}$$

$$g^{\mu\nu}g^{\rho\sigma}\bar{G}_{A^\nu A^\rho}\bar{G}_{A^\sigma A^\mu} = 3(\bar{G}_A)^2 + \xi^2(\bar{G}_\xi)^2 \tag{A.8}$$

$$\bar{G}_{A^0 A^0} = -(1 - \bar{\omega}_A^2/\bar{m}_A^2)\bar{G}_A - \xi(\bar{\omega}_\xi^2/\bar{m}_\xi^2)\bar{G}_\xi. \tag{A.9}$$

The first expression is needed for the first order result (the gauge boson loop), the second and third for the second order result (the gauge boson loop and the mixed gauge boson-Goldstone boson loop respectively).

For the one-loop calculation we only need the +-branch equal time propagator:

$$\bar{G}_i^{++}(0) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\bar{\omega}_i}, \tag{A.10}$$

where we used $\Theta(0) = 1/2$.

A.2 Dressed propagators

We will define dressed or resummed propagators, needed for the non-perturbative one-loop correction. The quadratic part of the potential, which has pieces in both $\mathcal{L}^{\rm free}$ and $\mathcal{L}^{\rm int}$, can be written in the form $\mathcal{L}^{\rm quad} = -(1/2)\sum_{i,j} \chi_i(x^\mu) K^{ij}(x^\mu) \chi_j(x^\mu)$. The dressed Green's functions are defined as for the free case (A.3), but now with possible time-dependent pieces in the wave operator K^{ij}. For the one-loop calculation we only need the (++)-propagator, which we discuss below; for ease of notation we drop the (++)-subscript.

The dressed Green's function satisfies the equation $K^{ij}(x^\mu) G_{jk}(x^\mu - y^\mu) = -i\delta(x^\mu - y^\mu)\delta_{ik}$. Fields with diagonal quadratic terms $K^{ij} \propto \delta^{ij}$ decouple from the other fields, and we can express the Green's function in terms of the mode functions in the usual way. For coupled fields, as is the case with A^0 and θ in our case, something similar is possible, but this involves more work. Consider a real scalar with canonical kinetic terms, then $K^{ii} = \Box + m_i^2$. Expand the field

$$\phi_i(x^\mu) = \int \frac{\mathrm{d}^3 k}{(2\pi^3)} \frac{1}{\sqrt{2\bar{\omega}_{\vec{k},i}}} \left[a_{\vec{k}} U_{\vec{k},i}(t) \mathrm{e}^{i\vec{k}\cdot\vec{x}} + a_{\vec{k}}^\dagger U_{\vec{k},i}^*(t) \mathrm{e}^{-i\vec{k}\cdot\vec{x}} \right], \tag{A.11}$$

with boundary conditions $U_{\vec{k},i}(0) = 1$, $\dot{U}_{\vec{k},i}(0) = -i\bar{\omega}_{\vec{k},i}$ such that $U_{\vec{k},i}$ is the positive frequency mode for scalar ϕ^i. The Fourier transform of the Green's function $G = \langle T(\phi(x^\mu)\phi(x'^\mu))\rangle$ can then be written in terms of the mode functions:

$$G_{\vec{k},i}(t,t') = \frac{1}{2\bar{\omega}_{\vec{k},i}} \Big(U_{\vec{k},i}(t) U_{\vec{k},i}^*(t') \Theta(t-t') + U_{\vec{k},i}(t') U_{\vec{k},i}^*(t) \Theta(t'-t) \Big). \tag{A.12}$$

The mode functions satisfy the wave equation with a time-dependent frequency:

$$K^{ii}(t,\vec{k}) U_{\vec{k},i}(t) = \left[\partial_t^2 + \omega_i^2(t)\right] U_{\vec{k},i}(t) = 0, \tag{A.13}$$

such that $G_i(x-x') = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} G_{\vec{k},i}(t,t')$ indeed satisfies the Green's function equation. To show this use that the Wronskian $\dot{U}_{\vec{k},i} U_{\vec{k},i}^* - U_{\vec{k},i} \dot{U}_{\vec{k},i}^* = -2i\bar{\omega}_{\vec{k},i}$ is constant in time.

For the one-loop calculation we only need the equal-time propagator which is

$$G_i(0) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{|U_{\vec{k},i}|^2}{2\bar{\omega}_{\vec{k},i}}. \tag{A.14}$$

A.3 Fermions

First we go to a field basis where the mass matrix is diagonal. For each fermionic field ψ the quadratic part of the Lagrangian can then be written as

$$\mathcal{L}_{\psi}^{(2)} = \bar{\psi} K \psi = \bar{\psi} \left[i\gamma^{\mu}\partial_{\mu} - m_{\psi} \right] \psi. \tag{A.15}$$

The dressed propagator is defined as $K(x)D_{\psi}(x-y) = i\delta(x-y)\mathbf{I}$, it is a Green's function of the Dirac operator. As usual we can expand the fermion field

$$\psi = \sum_{s} \int \frac{\mathrm{d}^3 k}{(2\pi)^3 \sqrt{2\bar{\omega}_{\vec{k}}}} \left[b_{\vec{k},s} u_{\vec{k},s} \mathrm{e}^{i\vec{k}\cdot\vec{x}} + d^{\dagger}_{\vec{k},s} v_{\vec{k},s} \mathrm{e}^{-i\vec{k}\cdot\vec{x}} \right], \tag{A.16}$$

with $\{b_{\vec{k},s}, b^{\dagger}_{\vec{k}',s'}\} = \{d_{\vec{k},s}, d^{\dagger}_{\vec{k}',s'}\} = (2\pi)^3\delta(\vec{k}-\vec{k}')\delta_{ss'}$. For a Majorana spinor we have $d_{\vec{k}} = b_{\vec{k}}$, i.e. a particle is its own anti-particle. The spinor function $u_{\vec{k},s}$ satisfies the equation $(i\partial_t - \mathcal{H}_{\vec{k}})u_{\vec{k},s} = 0$ with $\mathcal{H}_{\vec{k}} = \gamma^0(\gamma^i k_i + m_{\psi})$, the Fourier transformed Hamiltonian. Now we make the Ansatz

$$u_{\vec{k},s} = N \left[i\partial_t + \mathcal{H}_{\vec{k}} \right] U_{\psi}(\vec{k}) R_{s,u}, \qquad v_{\vec{k},s} = N \left[i\partial_t + \mathcal{H}_{-\vec{k}} \right] V_{\psi}(\vec{k}) R_{s,v}. \tag{A.17}$$

The spinors R_s are helicity eigenstates, normalized such that $R^{\dagger}_s R_{s'} = \delta_{ss'}$. Further $\gamma^0 R_{s,u} = R_{s,u}$ and $\gamma^0 R_{s,v} = -R_{s,v}$. The mode functions are each other's complex conjugates: $V^*_{\vec{k}} = U_{\vec{k}}$. Using usual free field normalization for the mode functions at $t=0$ gives $N = 1/\sqrt{\bar{\omega}_{\vec{k}} + \bar{m}_{\psi}}$ for the normalization factor. The mode function equation is

$$[\partial_t^2 + k^2 + m_{\psi}^2 - i\dot{m}_{\psi}]U_{\psi} = 0. \tag{A.18}$$

This is of the same form as the mode equation for the scalar field, namely a wave equation with time dependent frequency. Splitting the frequency in a time-independent and dependent part gives $\bar{\omega}_{\psi}^2 = k^2 + \bar{m}_{\psi}^2$ and $\delta\omega_{\psi}^2 = \delta m_{\psi}^2 - i\dot{m}_{\psi}$. It can be solved analogously to the scalar field case. Make the Ansatz

$$U_{\psi} = \mathrm{e}^{-i\bar{\omega}_{\psi}t}(1 + f_{\psi}), \qquad U_{\psi}(0) = 1, \;\; \dot{U}_{\psi}(0) = -i\bar{\omega}_{\psi}. \tag{A.19}$$

The dressed equal-time propagator is now

$$\begin{aligned} G_{\psi}(0) &= \langle \psi(t)\bar{\psi}(t) \rangle = \sum_{s} \int \frac{\mathrm{d}^3 k}{(2\pi^3)2\bar{\omega}_{\psi}} u_{\vec{k},s}\bar{u}_{\vec{k},s} \\ &= \sum_{\text{d.o.f.}} \frac{1}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[1 - \frac{\bar{\omega}_{\psi} - \bar{m}_{\psi}}{\bar{\omega}_{\psi}} |U_{\psi}|^2 \right]. \end{aligned} \tag{A.20}$$

The sum over the d.o.f. gives a factor 4 for a Dirac fermion, and a factor 2 for a Majorana/Weyl fermion.

B. Perturbative calculation

In this appendix we calculate the one-loop corrected equations of motion in arbitrary gauge to show explicitly that the results are gauge independent on-shell. The equations of motion in the CTP formalism are given by $\delta\Gamma/\delta\phi^+|_{J_\pm=0} = 0$. Diagrammatically this corresponds to all one-loop diagrams with one external h^+ leg. To isolate the divergent parts we need to go to second order in coupling.

B.1 First order

At zeroth order the tadpole diagram contributes, and we recover the classical equations of motion: $0 = i\Gamma_h^+ = i(-i(\Box\phi + V_\phi))$, see (3.9). We write

$$0 = \Box\phi + V_\phi + A_1 + A_2 + \text{finite} \tag{B.1}$$

with A_1, A_2 the first and second order contribution respectively (with one and two vertex insertions respectively). At first order four diagrams contribute, with $\{h, \theta, \eta, A^\mu\}$ running in the loop, and

$$\begin{aligned} A_1 &= \frac{i}{2}\Gamma^+_{h\alpha\alpha}\bar{G}^{++}_\alpha(0) \\ &= \frac{1}{2}\left[\left(\partial_\phi\delta m_h^2\right)\bar{G}_h^{++}(0) + \left(\partial_\phi\delta m_\theta^2\right)\bar{G}_\theta^{++}(0) - 2\left(\partial_\phi\delta m_\eta^2\right)\bar{G}_\eta^{++}(0) + \left(\partial_\phi\delta m^2_{A^\mu A^\nu}\right)\bar{G}^{++}_{A^\mu A^\nu}(0)\right]. \end{aligned} \tag{B.2}$$

The overall half factor is a symmetry factor, relating to the reflection symmetry of the Feynman diagrams. The two-point vertex is $\Gamma^+_{ii} = -i\delta m^2_{ii}$. The η loop picks up a minus sign because of the anti-commuting nature of η, and a factor two for the two fermionic degrees of freedom. The gauge boson term can be rewritten using (A.7):

$$(\partial_\phi\delta m^2_{A^\mu A^\nu})\bar{G}^{++}_{A^\mu A^\nu}(0) = (-\partial_\phi\delta m_A^2)g^{\mu\nu}\bar{G}^{++}_{A^\mu A^\nu}(0) = (\partial_\phi\delta m_A^2)(3\bar{G}_A^{++}(0) + \xi\bar{G}_\xi^{++}(0)). \tag{B.3}$$

Taking the large momentum limit, the equal time propagator (A.10) behaves as

$$\bar{G}_i^{++}(0) = \frac{1}{4\pi^2}\int k^2 \mathrm{d}k\left[\frac{1}{k} - \frac{1}{2}\frac{\bar{m}_i^2}{k^3} + \ldots\right] = \frac{1}{8\pi^2}\left[\Lambda^2 - \frac{1}{2}\bar{m}_i^2\ln\Lambda^2 + \text{finite}\right]. \tag{B.4}$$

Thus A_1 becomes (B.2):

$$\begin{aligned} A_1 &= \frac{\partial_\phi}{16\pi^2}\left[\delta m_h^2 + \delta m_\theta^2 - 2\delta m_\eta^2 + 3\delta m_A^2 + \delta m_\xi^2\right]\Lambda^2 \\ &\quad -\frac{1}{2}\left[\partial_\phi\delta m_h^2\bar{m}_h^2 + \partial_\phi\delta m_\theta^2\bar{m}_\theta^2 - 2\partial_\phi\delta m_\eta^2\bar{m}_\eta^2 + 3\partial_\phi\delta m_A^2\bar{m}_A^2 + \partial_\phi\delta m_\xi^2\bar{m}_\xi^2\right]\ln\Lambda^2. \end{aligned} \tag{B.5}$$

Here we defined $m_\xi^2 = \xi m_A^2$ analogous to (A.5). Upon inserting explicit mass terms, we infer that the quadratic divergence is gauge independent, but that the log-divergence depends on ξ. As we will see, this gauge dependence is cancelled by the second order term.

B.2 Second order

Consider first the diagonal loops with a single field running in the loop, and a three and two-point vertex insertion. The mixed loop, with propagators for both θ and A^0 is discussed afterwards. For each field running in the loop there are two diagrams that contribute, one with a $\Gamma^+_{\alpha\alpha}$ mass insertion and G^{++}_α propagators, and one with a $\Gamma^-_{\alpha\alpha}$ mass insertion and G^{+-}_α propagators. Let us start with the Higgs boson loop h. Its contribution to the equations of motion at second order is

$$\begin{aligned} A_2 \supset\ & \frac{i}{2}\int \mathrm{d}^4x'\Gamma^+_{hhh}(x)\Big[\bar{G}^{++}_h(x-x')\Gamma^+_{hh}(x')\bar{G}^{++}_h(x'-x) + \bar{G}^{+-}_h(x-x')\Gamma^-_{hh}(x')\bar{G}^{-+}_h(x'-x)\Big] \\ =\ & -\frac{i}{2}(\partial_\phi \delta m_h^2(t))\int \mathrm{d}^4x'\delta m_h^2(t')\left[\bar{G}^{++}_h(x-x')^2 - \bar{G}^{+-}_h(x-x')^2\right]. \end{aligned} \tag{B.6}$$

Here we used for the two-point function $\Gamma^+_{hh} = -\Gamma^-_{hh} = -i\delta m_h^2$. The overall symmetry factor 1/2 originates, again, from a reflection symmetry. Plugging in the expressions for the propagators gives

$$\begin{aligned} & \int \mathrm{d}^4x'\left[\bar{G}^{++}_h(x-x')^2 - \bar{G}^{+-}_h(x-x')^2\right] \\ & = \int \mathrm{d}^4x' \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\frac{\mathrm{d}^3p}{(2\pi)^3 2\omega_p}\Big[\mathrm{e}^{-i(k+p)(x-x')}\Theta(t-t') + \mathrm{e}^{-i(k+p)(x'-x)}\Theta(t'-t) - \mathrm{e}^{i(k+p)(x-x')}\Big] \\ & = \int \mathrm{d}t' \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{-2i}{(2\bar{\omega}_{\vec{k},h})^2}\Big[\sin[2\bar{\omega}_{\vec{k},h}(t-t')]\Theta(t-t')\Big]. \end{aligned} \tag{B.7}$$

To get to the second line, we used that integration over $\vec{x}'$ gives a factor $\delta^3(\vec{k}+\vec{p})$. This can be integrated over $\vec{p}$, which sets $\bar{\omega}_{\vec{k}} = \bar{\omega}_{\vec{p}}$. Putting it all together the h-loop contributes

$$A_2 \supset -\int \frac{\mathrm{d}^3k}{(2\pi)^3(2\bar{\omega}_{\vec{k},h})^2}\delta m_h^2(t)\int \mathrm{d}t'\delta m_h^2(t')\Big[\sin[2\bar{\omega}_{\vec{k},h}(t-t')]\Theta(t-t')\Big]. \tag{B.8}$$

To extract the divergent part we partially integrate:

$$\begin{aligned} \int \mathrm{d}t'\delta m^2(t')\sin(2\omega(t-t'))\Theta(t-t') &= \frac{\delta m^2(t')}{2\omega}\cos(2\omega(t-t'))\Big|_{t'=t_0}^{t'=t} - \int \mathrm{d}t'\frac{\delta\dot{m}^2(t')}{2\omega}\cos(2\omega(t-t')) \\ &= \frac{\delta m^2(t')}{2\omega} + \mathcal{O}(\omega^{-2}), \end{aligned} \tag{B.9}$$

where we set $\delta m^2(t_0) = 0$ at the initial time. And thus

$$A_2 \supset -(\partial_\phi \delta m_h^2(t))\delta m_h^2(t)\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{(2\bar{\omega}_{\vec{k},h})^3} + \mathcal{O}(\bar{\omega}_{\vec{k},h}^{-4}) = -\frac{1}{32\pi^2}(\partial_\phi \delta m_h^2)\delta m_h^2 \ln \Lambda^2 + \text{finite}. \tag{B.10}$$

In the last step we expanded in large $|\vec{k}|$.

The calculation of the θ and η loops proceeds analogously, and gives a contribution just as (B.10) with the appropriate mass; in addition the η-loops picks up an overall factor (-2) because of the two anti-commuting d.o.f. The contribution for the gauge field is

$$
\begin{aligned}
A_2 &\supset -\frac{i}{2}\partial_\phi \delta m_A^2(t) \int \mathrm{d}^4x' \delta m_A^2(t') g^{\mu\rho} g^{\nu\sigma} \left[\bar{G}^{++}_{A^\mu A^\nu}\bar{G}^{++}_{A^\rho A^\sigma} - \bar{G}^{+-}_{A^\mu A^\nu}\bar{G}^{-+}_{A^\rho A^\sigma}\right] \\
&= -\frac{i}{2}\partial_\phi \delta m_A^2(t) \int \mathrm{d}^4x' \delta m_A^2(t') \left[3\left((\bar{G}_A^{++})^2 - (\bar{G}_A^{+-})^2\right) + \xi^2\left((\bar{G}_\xi^{++})^2 - (\bar{G}_\xi^{+-})^2\right)\right] \\
&= -\frac{1}{32\pi^2}\left[3(\partial_\phi \delta m_A^2)\delta m_A^2 + (\partial_\phi \delta m_\xi^2)\delta m_\xi^2\right] \ln\Lambda^2 + \text{finite}. \qquad \text{(B.11)}
\end{aligned}
$$

In the first line we used the definition of mass (3.8) $m_{A^\mu A^\nu} = -g^{\mu\nu} m_A^2$ with $m_A^2 = g^2\phi^2$. To get the second line we used (A.8). The expression has been reduced to a sum of two scalar integrals, which result in expressions analogous to (B.10) to give the final result, given in the last line above. Adding it all up gives

$$
\begin{aligned}
A_2^{\text{diag}} = -\frac{1}{32\pi^2}\Big[& (\partial_\phi \delta m_h^2)\delta m_h^2 + (\partial_\phi \delta m_\theta^2)\delta m_\theta^2 - 2(\partial_\phi \delta m_\eta^2)\delta m_\eta^2 + 3(\partial_\phi \delta m_A^2)\delta m_A^2 \\
& + (\partial_\phi \delta m_\xi^2)\delta m_\xi^2\Big] \ln\Lambda^2. \qquad \text{(B.12)}
\end{aligned}
$$

In $\mathcal{L}^{\text{int}}$ there is also a derivative interaction mixing the gauge and the Goldstone boson. This leads to a mixed loop diagram. Since $\phi(t)$ does not depend on spatial coordinates, the derivatives will only act on time, and thus the mass terms contain factors g^{00}. The mixed diagram contributes

$$
\begin{aligned}
A_2^{\text{mix}} &= -i\partial_\phi \delta m_{A\theta}^2(t) \int \mathrm{d}^4x' \delta m_{A\theta}^2(t') g^{0\mu} g^{0\nu} \left[\bar{G}^{++}_{A^\mu A^\nu}\bar{G}^{++}_\theta - \bar{G}^{+-}_{A^\mu A^\nu}\bar{G}^{-+}_\theta\right] \qquad \text{(B.13)} \\
&= i\partial_\phi \delta m_{A\theta}^2(t) \int \mathrm{d}^4x' \delta m_{A\theta}^2(t') \left[\left(1 - \frac{\bar{\omega}_A^2}{\bar{m}_A^2}\right)\bar{G}_A^{++} + \frac{\xi\bar{\omega}_\xi^2}{\bar{m}_\xi^2} G_\xi^{++}\right] G_\theta^{++} - (++ \to +-).
\end{aligned}
$$

There is no symmetry factor 1/2 since there is no reflection symmetry. In the first line we used $\Gamma^+_{A^\mu\theta} = i\delta m_{A\theta}^2 g^{0\mu}$, and $m_{A\theta}^2 = \delta m_{A\theta}^2$. Using (A.9) we reduced the propagators to scalar propagators as before. Plugging in the explicit expressions we find

$$
\begin{aligned}
A_2^{\text{mix}} &= 2\partial_\phi \delta m_{A\theta}^2(t) \int \mathrm{d}t' \delta m_{A\theta}^2(t') \int \frac{\mathrm{d}^3k}{(2\pi)^3}\Bigg\{\Big[\frac{1 - \bar{\omega}_A^2/\bar{m}_A^2}{4\bar{\omega}_\theta\bar{\omega}_A}\sin((\bar{\omega}_A + \bar{\omega}_\theta)\Delta t) \\
&\quad + \frac{\xi\bar{\omega}_\xi^2/\bar{m}_\xi^2}{4\bar{\omega}_\theta\bar{\omega}_\xi}\sin((\bar{\omega}_\xi + \bar{\omega}_\theta)\Delta t)\Big]\Theta(\Delta t)\Bigg\} \\
&= 2(\partial_\phi \delta m_{A\theta}^2)\delta m_{A\theta}^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1 - \bar{\omega}_A^2/\bar{m}_A^2}{4\bar{\omega}_\theta\bar{\omega}_A(\bar{\omega}_\theta + \bar{\omega}_A)} + \frac{\xi\bar{\omega}_\xi^2/\bar{m}_\xi^2}{4\bar{\omega}_\theta\bar{\omega}_\xi(\bar{\omega}_\theta + \bar{\omega}_\xi)}\right] + \mathcal{O}(\omega_i^{-4}) \\
&= \frac{(3+\xi)}{64\pi^2}(\partial_\phi \delta m_{A\theta}^2)\delta m_{A\theta}^2 \ln\Lambda^2 + \text{finite}. \qquad \text{(B.14)}
\end{aligned}
$$

In the second step we performed a partial integration to isolate the divergent parts.

The one-loop correction to the equation of motion is $A_1 + A_2^{\rm diag} + A_2^{\rm mix}$, which gives

$$
\begin{aligned}
0 &= \Box\phi + V_\phi + \frac{\partial_\phi}{16\pi^2}\left[\delta m_h^2 + \delta m_\theta^2 - 2\delta m_\eta^2 + 3\delta m_A^2 + \delta m_\xi^2\right]\Lambda^2 \\
&- \frac{1}{32\pi^2}\left[(\partial_\phi m_h^2)m_h^2 + (\partial_\phi m_\theta^2)m_\theta^2 - 2(\partial_\phi m_\eta^2)m_\eta^2 + 3(\partial_\phi m_A^2)m_A^2 + (\partial_\phi m_\xi^2)m_\xi^2\right]\ln\Lambda^2 \\
&+ \frac{(3+\xi)}{64\pi^2}(\partial_\phi m_{A\theta}^2)m_{A\theta}^2 \ln\Lambda^2
\end{aligned}
\tag{B.15}
$$

where we used $\partial_\phi \delta m_\alpha^2 = \partial_\phi m_\alpha^2$.

Integrating to get the effective action gives:

$$
\begin{aligned}
\Gamma^{1-\rm loop} &= -\frac{1}{16\pi^2}\int {\rm d}^4x \Big\{ \left[m_h^2 + m_\theta^2 - 2m_\eta^2 + 3m_A^2 + m_\xi^2\right]\Lambda^2 \\
&\qquad -\frac{1}{4}\left[m_h^4 + m_\theta^4 - 2m_\eta^4 + 3m_A^4 + m_\xi^4 - \frac{1}{2}(3+\xi)m_{\theta A}^4\right]\ln\Lambda^2 \Big\} \\
&= -\frac{1}{16\pi^2}\int {\rm d}^4x \left[\Lambda^2\left(V_{hh} + V_{\theta\theta} + 3m_A^2\right) - \frac{\ln\Lambda^2}{4}\left(V_{hh}^2 + V_{\theta\theta}^2 + 3m_A^4 - 6V_{\theta\theta}m_A^2\right)\right].
\end{aligned}
\tag{B.16}
$$

plus finite and field-independent terms. The gauge parameter ξ cancels, and gauge invariance of the final result is manifest, provided we use the zeroth order equation of motion (together with gauge invariance) to write $\delta m_{A\theta}^4 = 4m_A^2 V_{\theta\theta}$ (3.32). To get the final result we inserted the explicit form of the masses from (3.8), and the definition $m_\xi^2 = \xi m_A^2$. This result is in agreement with the non-perturbative calculation (3.31). The gauge dependent part of m_θ^2 and m_A^2 cancels against that of the ghosts, i.e. $(m_\theta^2 + m_\xi^2 - 2m_\eta^2) = V_{\theta\theta}$, making the quadratic terms coming from the first order calculation gauge invariant. Combining the first and second order calculation renders also the logarithmic divergences gauge invariant, but only upon using the equations of motion (3.32): $(m_\theta^4 - 2m_\eta^4 + m_\xi^4 - 2\xi V_{\theta\theta}m_A^2) = (V_{\theta\theta})^2$.

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